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Circular regions under uniform pressures in second order elasticity

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Three problems concerning isotropic, compressible materials have been solved. These are (i) an infinite isotropic, compressible medium with a circular hole subjected to a uniform internal pressure, (ii) an isotropic, compressible finite disc subjected to uniform compsession at the boundary and (iii) an elastic circular misfitting inhomogeneity in an infinite isotropic, compressible matrix. Second order elasric effects in the elastic fields for all the three cases have been evaluated in terms of complex coordinates in the initial configuations. The results obtained reveal the influence of the elastic properties of the materials to a greater extent than those obtained in the case of infinitesimal elasticity. Theer problems also illustrate the very important feature of second order elasticity in which the undeformed and deformed configurations are different from each other.

INTRODUCTION

The complex variable methods, as developed by Muskhelishvili (1963), and others have been usefully employed in solving problems in infinitesimal elasticity. The technique mainly consists in expressing the entire elastic field in terms of a scalar function ϕ, called the Airy's stress function. This function is expressible in terms of a set of two potential functions which are evaluated with the help of given conditions of the problem. For the second order elasticity case Airy's stress function is expanded as power series of a parameter which depends upon the physical conditions of the problem. This leads to an additional set of two potential functions. The earlier works of Adkins *et al* (1953, 1954) and more recent works of Bhargava & Pande (1964, 1966, 1967) have developed some techniques of evaluating these sets of potential function. The techniques suggested by Adkins *et al* have limited applications. Bhargava & Pande, having developed the techniques, have only concerned themselves with the deformed state of the body ignoring the undeformed state. In the present paper the problems have been solved using the configuration. The work of this paper and that of Bhargava & Pande clearly establishes the fact that in second order elasticity the initial and final configurations are distinct from each other unlike the case of infinitesimal elasticity. A brief account of the mathematical preliminaries is essential and is given below.

Let the point $P_0(x_1, x_2)$ of a body B_0 in the undeformed state be displaced to $P(y_1, y_2)$ of B, the body in the deformed state, when referred to a fixed, plane Cartesian frame of reference.

In complex coordinates let

$$\left.\begin{aligned} \zeta &= x_1 + i\,x_2 \quad \text{and } z = y_1 + iy_2 \\ \text{thus,} \quad \bar{\zeta} &= x_1 - ix_2 \quad \text{and } \bar{z} = y_1 - i\,y_2 \end{aligned}\right\} \tag{1}$$

Let the displacement vector $D(u_1, u_2)$ be the complex function

$$D = u_1 + i\,u_2 \tag{2}$$

Hence,

$$z = \zeta + D \quad \text{and} \quad \bar{z} = \bar{\zeta} + \bar{D} \tag{3}$$

In the absence of body forces the equilibrium equations are satisfied if

$$T^{11} = T^{-22} = -4\,\frac{\partial^2\phi}{\partial\zeta^2} \quad \text{and } T^{12} = 4\,\frac{\partial^2\phi}{\partial\zeta\partial\bar{\zeta}} \tag{4}$$

where $T^{\alpha\beta}$ are the complex stress components in deformed state and

$$T^{11} = T^{-22} = p_{x_1x_1} - p_{x_2x_2} + 2i\,p_{x_1x_2} \text{ and } T^{-12} = p_{x_1x_1} + p_{x_2x_2} \tag{5}$$

$p_{\alpha\beta}$ being the stress components in Cartesian coordinates. When the resultant force along the boundary is zero,

$$\frac{\partial\phi}{\partial\zeta} = \frac{\partial\phi}{\partial\bar{\zeta}} = 0 \tag{6}$$

The Airy's stress function ϕ and the displacement function D, being analytic functions of ζ and $\bar{\zeta}$, may be expanded in powers series of a real parameter ϵ. Thus

$$\phi(\zeta,\bar{\zeta}) = \mu\epsilon\,[\,\phi_0(\zeta,\bar{\zeta}) + \epsilon\phi_1(\zeta,\bar{\zeta}) + \epsilon^2\phi_2(\zeta,\bar{\zeta}) + \cdots] \tag{7}$$

$$D(\zeta,\bar{\zeta}) = \epsilon\,[\,D_0(\zeta,\bar{\zeta}) + \epsilon D_1(\zeta,\bar{\zeta}) + \epsilon^2 D_2(\zeta,\bar{\zeta}) + \cdots] \tag{8}$$

The parameter ϵ is determined by the physical conditions of the problem. The first term of the expansions (7) and (8), namely ϕ_0 and D_0, pertain to the infinitesimal elasticity case. These are completely determined in terms of the potential functions $\Omega(\zeta)$, $\omega(\zeta)$. The first two terms jointly give the second order effects in the elastic field. The functions ϕ_1 and D_1 are determinable in terms of two sets of potential functions $\{\Omega(\zeta), \omega(\zeta)\}$ and $\{\Delta(\zeta), \delta(\zeta)\}$.

Since only second order effects are being considered, the functions ϕ_0, D_0, ϕ_1, D_1, alone in the expansions (7) and (8) are valid in the present case.

It is known that (Gresen & Zerna 1954),

$$\phi_0(\zeta,\bar{\zeta}) = \bar{\zeta}\Omega(\zeta) + \zeta\bar{\Omega}(\bar{\zeta}) + \omega(\zeta) + \bar{\omega}(\bar{\zeta}), \tag{9}$$

$$D_0(\zeta,\bar{\zeta}) = \kappa\,\Omega(\zeta) - \zeta\,\bar{\Omega}'(\bar{\zeta}) - \bar{\omega}'(\bar{\zeta}) \tag{10}$$

where $\kappa=(\lambda+3\mu)/(\lambda+\mu)$ for the plane strain and $\kappa=(5\lambda+6\mu)/(3\lambda+2\mu)$ for the plane stress, λ and μ being the Lame's constants.

The functions $\triangle(\zeta)$ and $\delta(\zeta)$ can be evaluated from the equation

$$\frac{\partial\phi_1}{\partial\bar{\zeta}} = \triangle(\zeta) + \zeta\triangle'(\bar{\zeta}) + \bar{\delta}'(\bar{\zeta}) + (\gamma-1)\,\Gamma(\zeta,\bar{\zeta})$$

$$+ (B_3/\kappa)\bar{\Omega}'(\bar{\zeta})D_o(\zeta,\bar{\zeta}) + k_1\int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\bar{\omega}''(\bar{\zeta})d\bar{\zeta}$$

$$+\kappa_2\int^{\bar{\zeta}}[\Omega'(\zeta)]^2 d\zeta + k_3\zeta[\bar{\Omega}'(\bar{\zeta})]^2 \tag{11}$$

The displacement function $D_1(\zeta,\bar{\zeta})$ is given by

$$D_1(\zeta,\bar{\zeta}) = \kappa\,\triangle(\zeta) - \zeta\bar{\triangle}'(\bar{\zeta}) - \bar{\delta}'(\bar{\zeta}) - (\gamma-1)\Lambda(\zeta,\bar{\zeta})$$

$$- (B_3/\kappa)\bar{\Omega}'(\bar{\zeta})D_0(\zeta,\bar{\zeta}) + k'_1\int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\bar{\omega}''(\bar{\zeta})d\bar{\zeta}$$

$$+ k'_2\int^{\bar{\zeta}}[\Omega'(\zeta)]^2 d\zeta + k'_3\zeta[\bar{\Omega}'(\bar{\zeta})]^2 \tag{12}$$

where
$$\Gamma(\dot{\zeta},\bar{\zeta}) = [\zeta\bar{\Omega}''(\bar{\zeta}) + \bar{\omega}''(\bar{\zeta})]\,[\bar{\zeta}\,\Omega'(\zeta) + \omega'(\zeta) - \kappa\bar{\Omega}(\bar{\zeta})]$$

$$+ [\Omega'(\zeta) + \bar{\Omega}'(\bar{\zeta})]\,[\zeta\bar{\Omega}'(\bar{\zeta}) + \bar{\omega}'(\bar{\zeta}) - \kappa\Omega(\zeta)] \tag{13}$$

and
$$\Lambda(\zeta,\bar{\zeta}) = [\zeta\bar{\Omega}''(\bar{\zeta}) + \bar{\omega}''(\bar{\zeta})]\,[\bar{\zeta}\Omega'(\zeta) + \omega'(\zeta) - \kappa\,\bar{\Omega}(\bar{\zeta})]$$

$$- [\kappa\Omega'(\zeta) - \bar{\Omega}'(\bar{\zeta})]\,[\zeta\bar{\Omega}'(\bar{\zeta}) + \bar{\omega}'(\bar{\zeta}) - \kappa\Omega(\zeta)] \tag{14}$$

The constants involved in (11) and (12) have been defined in Appendix. Further, of the two sets of constants k_i and k_i' in (11) and (12) respectively, one may be taken to be zero and the other be evaluated with the help of equations given in the Appendix. For example k_i may taken to be zero in the case of first boundary value problem while in the case of the second boundary value problem it is useful to take k_i' to be zero.

To ensure uniqueness and single-valuedness of ϕ and D the functions $\Omega(\zeta),\omega(\zeta),\triangle(\zeta)$ and $\delta(\zeta)$ must satisfy the following conditions :

$$[\Omega'(\zeta)]_c = 0;\ [\bar{\omega}''(\bar{\zeta})]_c = 0;\ [\kappa\Omega(\zeta) - \bar{\omega}'(\bar{\zeta})]_c = 0 \tag{15}$$

$$[\triangle'(\zeta)]_c = 0;\ [\bar{\delta}''(\bar{\zeta})]_c = 0;$$

$$[\kappa\triangle(\zeta) - \bar{\delta}'(\bar{\zeta})]_c = -[k_1'\int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\,\bar{\omega}''(\bar{\zeta})\,d\bar{\zeta} + k'_2\int^{\zeta}\Omega'(\zeta)\}^2 d\zeta]_c = 0 \tag{16}$$

where $[\quad]_c$ denotes the cyclic change in the function within the parenthesis while going once round the contour C lying entirely within the deformed state of the body.

With the evaluation of the functions $\Omega(\zeta)$, $\omega(\zeta)$, $\triangle(\zeta)$, $\delta(\zeta)$ and the determination of ϵ from the physical conditions the equations (7) - (12) along with (4) and (5) determine the elastic field completely in terms of Cartesian coordinates in the undeformed state of the body.

Problem 1. Infinite compressible medium with a circular hole under uniform radial pressure.

Let a homogeneous, isotropic, compressible infinite medium have a circular hole in it. Let the equation of the circlar boundary in complex coordinates be

$$|\zeta| = R \tag{17}$$

Further let this circular boundary be subjected to a uniform radial pressure P. Due to the pressure, deformations would set in. It is proposed to investigate the stress and displacement fields in the medium with the help of the technique outlined above.

It is known that the transformation

$$\zeta = \frac{R}{\zeta_1} \tag{18}$$

maps the region interior to the circular hole onto the region $|\zeta_1| > 1$ and the region $|\zeta| > R$ is mapped onto the region $|\zeta_1| < 1$. The circular boundary coincides with the boundary of the unit circle $|\sigma| = 1$ where σ is the value of ζ_1 on the boundary of the unit circle.

Obviously, $\qquad \sigma = e^{-i\theta}$ (19)

θ, $Re^{i\theta_1}$ measured positive in the clockwise direction, maps the boundary $\zeta = Re^{i\theta_1}$, θ_1 measured positive in the counter clockwise direction. With the introduction of the transformation (18), the functions Ω, ω, $\triangle$ and δ will become functions of ζ_1. Thus

$$\left.\begin{aligned}
\Omega(\zeta) &= \Omega\left(\frac{R}{\zeta_1}\right) = \Omega_1(\zeta_1),\\
\omega(\zeta) &= \omega\left(\frac{R}{\zeta_1}\right) = \omega_1(\zeta_1),\\
\triangle(\zeta) &= \triangle\left(\frac{R}{\zeta_1}\right) = \triangle_1(\zeta_1),\\
\delta(\zeta) &= \delta\left(\frac{R}{\zeta_1}\right) = \delta_1(\zeta_1).
\end{aligned}\right\} \tag{20}$$

The values of Ω_1 and ω_1 are known (Sokolnikoff 1956). They are

$$\left.\begin{aligned}
\Omega_1(\zeta_1) &= 0\\
\omega_1'(\zeta_1) &= -\frac{PR\zeta_1}{2}
\end{aligned}\right\}. \tag{21}$$

Choosing $\epsilon = \frac{1}{\mu}$, and taking $k_1 = k_2 = 0$, the equation (11) on the boundary of the unit circle yields

$$\Delta_1(\sigma) - \frac{1}{\sigma_3} \bar{\Delta}_1'(\bar{\sigma}) + \bar{\delta}_1'(\bar{\sigma}) + (\gamma-1)\Gamma(\sigma,\bar{\sigma})$$

$$+ (B_3 \mid \kappa)\bar{\Omega}'(\bar{\sigma})D_0(\sigma,\bar{\sigma}) + k_3 \frac{R}{\sigma} \left\{ \Omega'(\bar{\sigma}) \right\}^2 = 0 \tag{22}$$

Making appropriate substitutions, the equation (22) becomes

$$\Delta_1(\sigma) - \frac{1}{\sigma^3} \bar{\Delta}_1'(\bar{\sigma}) + \bar{\delta}_1'(\bar{\sigma}) = \frac{(\gamma-1)P^2R}{4\sigma} \tag{23}$$

On integration, the values of $\Delta_1(\zeta_1)$ and $\delta_1'(\zeta_1)$ are found to be

$$\left.\begin{aligned} \Delta_1(\zeta_1) &= 0. \\ \delta_1'(\zeta_1) &= \frac{(\gamma-1)P^2R\zeta_1}{4} \end{aligned}\right\} \tag{24}$$

Thus $\Omega(\zeta)$,$\omega(\zeta)$,$\Delta(\zeta)$,$\delta(\zeta)$ and ϵ being known, complete stress fiield is obtained from equations (4), (7), (9) and (11). Thus, making appropriate substitutions and changing into polar coordinates the stresses at any point in the medium are found to be

$$p_{rr} = \frac{-PR^2}{r^2} + \frac{(\gamma-1)P^2R^2}{2\mu r^2} \left[\frac{R^2}{r^2} - \left(\frac{2R^2}{r^2} - 1 \right) \right] \tag{25}$$

$$p_{\theta\theta} = \frac{PR^2}{r^2} + \frac{(\gamma-1)P^2R^2}{2\mu r^2} \left[\frac{R^2}{r^2} + \left(\frac{2R^2}{r^2} - 1 \right) \right], \tag{26}$$

$$p_{r\theta} = 0. \tag{27}$$

The displacement field is obtained directly by making proper substitutions in (8), (10), and (12). On changing into polar coordinates the displacements are

$$\left.\begin{aligned} u_r &= \frac{PR^2}{2\mu r} \left\{ 1 - \frac{(\gamma-1)P}{2\mu} \left(1 - \frac{R^2}{r^2} \right) \right\} \\ u_\theta &= 0. \end{aligned}\right\}. \tag{28}$$

An examination of the above results would prove that in the case of infinitesimal elasticity stresses are free from the Poisson's ratio but in the second order case, the Poisson's ratio does affect the hoop stress as is but natural. Secondly, though the second order effects for the displacements are zero at the inner boundary but at other points they are significant.

PROBLEM 2. CIRCULAR DISC UNDER UNIFORM COMPRESSION

Let a circular disc of homogeneous, isotropic compressible material and of radius a be subjected to a uniform compressible force Q along its boundary. Let the circular edge have the equation

$$|\zeta| = R_0 \tag{29}$$

in complex coordinates in the undeformed state. Due to compression along the boundary, deformations would set in. It is proposed to obtain the stresses and displacements in the disc due to deformation.

The transformation

$$\zeta = R_0 \zeta_1 \tag{30}$$

gives an isomorphic mapping of $|\zeta| = R_0$ onto the unit circle $|\sigma| = 1$ where σ is the value of ζ_1 on the boundary of the unit circle.

It is known (Sokolnikoff 1956) that

$$\Omega(\zeta) = -\frac{Q\zeta}{4}, \quad \omega'(\zeta) = 0 \tag{31}$$

Taking $\epsilon = 1/\mu$ and following the procedure adopted in problem 1 solved above, the functions $\triangle(\zeta)$, $\delta'(\zeta)$ are found to be

$$\triangle(\zeta) = [2(\gamma-1)(\kappa-1) + (B_1 - B_3)] \frac{Q^2\zeta}{32}, \quad \delta'(\zeta) = 0 \tag{32}$$

The functions $\Omega(\zeta)$, $\omega(\zeta)$, $\triangle(\zeta)$, $\delta(\zeta)$ being evaluated and ϵ being known in the medium, the stresses at any point in the medium in polar coordinates are found to be

$$\left.\begin{aligned} p_{rr} &= -Q \\ p_{\theta\theta} &= -Q \\ p_{r\theta} &= 0 \end{aligned}\right\} \tag{33}$$

Similarly the displacements are

$$\left.\begin{aligned} u_r &= -\frac{QR_0}{4\mu}\left[(\kappa-1) - \frac{(\kappa+1)QB_1}{8\mu}\right] \\ u_\theta &= 0 \end{aligned}\right\}. \tag{34}$$

It will be seen that though at the boundary, for the stresses $p_{rr} = p_{\theta\theta} = -Q$, but at any point the pressures significantly differ from those in the case of classical elasticity. The same is true for displacements.

PROBLEM 3. CIRCULAR INHOMOGENEITY IN AN INFINITE ELASTIC MEDIUM

Using the results found above a very important problem obtaining in various technological and metallurgical processes can be solved. It is the problem of circular inhomogeneity in an infinite elastic matrix. Let an isotropic, elastic circular disc of radius $a(1+\delta)$, δ being within elastic limits, be squeezed and inserted into another isotropic elastic, infinite medium with a circular hole of radius a. Let the infinite medium, called matrix, be of a different material. Due to pressure of the matrix the inserted disc, called inclusion, does not attain the free configuration and the boundaries of both the inclusion and the matrix deform. On physical grounds, the equilibrium boundary will be a concentric circle of radius, say $a(1+\epsilon_o)$, ϵ_o also being within elastic limits. It is supposed that no relative slipping takes place along the equilibrium boundary. The whole system may be taken to be welded along the equilibrium boundary. Due to deformations, stresses and strains will develop in the bodies. Let p be the common pressure along the equilibrium interface. The solution lies in evaluating ϵ_o which gives the equilibrium position of the system. The equilibrium pressure and the hoop stresses have been calculated.

In the case of inclusion the displacements along the equilibrium boundary are given by equation (34). They are

$$u_{ri} = -\frac{pa(1+\epsilon_o)}{4\mu_i}\left[(\kappa_i-1) - \frac{(\kappa_i+1)pB_{1i}}{8\mu_i} \right], \; u_{\theta i} = 0. \tag{35}$$

For the matrix, equation (28) gives

$$u_{rm} = \frac{pa(1+\epsilon_o)}{2\mu_m}, \; u_{\theta m} = 0. \tag{36}$$

In the present problem $u_{ri} = -a(\delta-\epsilon_o)$ and $u_{rm} = a\epsilon_o$

Hence,
$$u_{ri} = -a(\delta-\epsilon_o) \tag{37}$$
$$u_{rm} = a\epsilon_o. \tag{38}$$

Eliminating p from (35) and (36), the following quadratic equation in ϵ_o is obtained.

$$\begin{aligned}&[\, B_{1i}(\kappa_i+1)\, \mu_m^2 - 4\mu_i\mu_m(\kappa_i-1) - 8\mu_i^2 \,]\, \epsilon_o^2 \\ &- [\, 4\mu_i\mu_m(\kappa_i-1) + 8\mu_i^2(1-\delta) \,]\epsilon_o + 8\mu_i^2\delta = 0\end{aligned} \tag{39}$$

This gives two values of ϵ_o One of these values is inadmissible on physical grounds becasuse it makes the value of ϵ_o less than -1 which means that radius of the equilibrium interface is negative. Thus ϵ_o is found to be

$$\epsilon_o = \frac{\begin{aligned}&-\{\, 2\mu_i^2\,(1-\delta) + \mu_i\mu_m\,(\kappa_i-1) \,\} \\ &\quad + [\{2\mu_i^2\,(1-\delta) + \mu_i\mu_m\,(\kappa_i-1)\,\}^2 \\ &\quad + 4\mu_i^2\delta\,\{4\mu_i^2 + 2\mu_i\mu_m\,(\kappa_i-1) + \mu_m^2 A \,\}]^{1/2}\end{aligned}}{4\mu_i^2 + 2\mu_i\mu_m\,(\kappa_i-1) + \mu_m^2 A.} \tag{40}$$

where $A = -\frac{1}{2} B_{1i}(\kappa_i + 1)$

Substituting for ϵ_o in (38), p is found to be

$$p = \frac{3\mu_m \epsilon_o}{1+\epsilon_o} . \tag{41}$$

Further $p_{\theta\theta}$ for the inclusion and the matrix are found from equations (33), (36) after substituting for R_o, R and ϵ_o. Thus

$$p_{\theta\theta i} = -p \tag{42}$$

$$p_{\theta\theta m} = p\{1+(\gamma-1)\frac{p}{\mu}\} . \tag{43}$$

The jump in hoop stresses is found to be

$$p_{\theta\theta m} - p_{\theta\theta i} = p\{2+(\gamma-1)\frac{p}{\mu}\} . \tag{44}$$

The case when the inclusion and the matrix are of the same material can be obtained as special case from equation (40)—(44) by putting $\mu_i = \mu_m$.

Thus $$\epsilon_o = \frac{-\{(\kappa+1)-2\delta\} + [\{(\kappa+1)-2\delta\}^2 + 4\delta\{2(\kappa+1)+A\}]^{1/2}}{2(\kappa+1)+A} \tag{45}$$

Similarly for the linear elasticity case

$$\epsilon_o = \frac{(\lambda+\mu)}{(\lambda+2\mu)}\delta . \tag{46}$$

It will be found that the second order effects increase the radius of the equilibrium boundary and equilibrium pressure is reduced as compared to the infinitesimal case, which is consistent on the physical grounds. Hoop stress also diminishes in the second order case. The cases of rigid inclusion in an elastic matrix and elastic inclusion in a rigid matrix can be deduced from the results obtained here. In the former case $\epsilon_o = \delta$ and in the latter case $\epsilon_o = 0$. Again taking λ, μ to be zero, all the stresses and displacements are found to be zero in the case of cavity.

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Appendix

Elastic constants involved in the problems.

$$\left.\begin{aligned} C_1 &= -\frac{2}{\mu}\left[\frac{\partial^2 W'}{\partial J_1^2}\right]_o ; \; C_2 = -\frac{2}{\mu}\left[\frac{\partial^2 W'}{\partial J_1 \partial J_2}\right]_o \\ C_3 &= -\frac{2}{\mu}\left[\frac{\partial W'}{\partial J_3}\right]_o ; \; C_4 = -\frac{2}{\mu}\left[\frac{\partial^3 W'}{\partial J_1^3}\right]_o \end{aligned}\right\} \tag{A-1}$$

wher $[\;]_o$ indicates the values of the function within the bracket when $J_r = 0$.

$$\left.\begin{aligned} B_1 &= \frac{6C_1 - 4C_2 - 7}{2C_1 - 1} \\ B_2 &= \frac{(2C_1 - 1)(4C_1^2 - 8C_1 + 1) - 12C_3 - 8C_4}{(2C_1 - 1)^3} \end{aligned}\right\} \tag{A-2}$$

$$\left.\begin{aligned} & B_1' = B_1 - (\kappa+1);\ B_1'' = \tfrac{1}{2} B' + B_1 \\ & B_2' = B_2 - \tfrac{1}{2} (\kappa+1)^2;\ B_3 = B_1 - \frac{2B_2}{\kappa+1} \\ & B_4 = \tfrac{1}{2} B' - B'_2 ;\ \gamma = \frac{B_1}{\kappa+1} \end{aligned}\right\} \tag{A-3}$$

$$\left.\begin{aligned} & k_1 + k_1' = B_1' ;\ k_3 - k_2 = \frac{B_3}{\kappa} - B_1 \\ & \kappa k_2 - k_2' = B_4 ;\ \kappa k_3 + k_2' = \kappa B_1'' - B_3 - B_4 \end{aligned}\right\} \tag{A-4}$$

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